Vector fields on Π-symmetric flag supermanifolds[1]

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Abstract

The main result of this paper is the computation of the Lie superalgebras of holomorphic vector fields on the complex Π-symmetric flag supermanifolds, introduced by Yu.I. Manin. We prove that with one exception any vector field is fundamental with respect to the natural action of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$.

1 Introduction

A Π-symmetric flag supermanifold is a subsupermanifold in a flag supermanifold in $\mathbb{C}^{n|n}$ that is invariant with respect to an odd involution in $\mathbb{C}^{n|n}$. This supermanifold possesses a transitive action of the linear classical Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$, which belongs to one of two "strange series" in the Kac classification [Kac]. It turns out that with one exceptional case all global holomorphic vector fields are fundamental for this action of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$. In the simplest case of super-Grassmannians the similar result was obtained in [Oni].

The main result of this paper was announced in [V4] and the idea of the proof was given in [V2]. The goal of this notes is to give a detailed proof. We also describe the connected component of the automorphism supergroup of this supermanifolds.

2 Flag supermanifolds

We will use the word "supermanifold" in the sense of Berezin and Leites [BL], see also [Oni] for details. Throughout, we will restrict our attention to the complex-analytic version of the theory of supermanifolds. Recall that a *complex-analytic superdomain of dimension* $n|m$ is a $\mathbb{Z}_2$-graded ringed space of the form $(\mathcal{U}_0, \mathcal{F}_{\mathcal{U}_0} \otimes_{\mathbb{C}} \bigwedge(m))$, where $\mathcal{F}_{\mathcal{U}_0}$ is the sheaf of holomorphic functions on an open set $\mathcal{U}_0 \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators. A *complex-analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$. Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ be a supermanifold and $\mathcal{J}$ be the subsheaf of ideals generated by odd elements in $\mathcal{O}_{\mathcal{M}}$. We set $\mathcal{F}_{\mathcal{M}_0} := \mathcal{O}_{\mathcal{M}}/\mathcal{J}$. Then $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}_0})$ is a usual complex-analytic manifold, it is called the *underlying space* of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}_0})$.

In this paper we denote by $\mathbf{F}^{m|n}_{k|l}$ a flag supermanifold of type $k|l$ in the vector superspace $\mathbb{C}^{m|n}$. Here we set $k = (k_1, \dots, k_r)$ and $l = (l_1, \dots, l_r)$ such that

$$\begin{gathered} 0 \leq k_r \leq \ldots \leq k_1 \leq m, \quad 0 \leq l_r \ldots \leq l_1 \leq n \quad \text{and} \\ 0 < k_r + l_r < \ldots < k_1 + l_1 < m + n. \end{gathered} \tag{1}$$

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A Π-symmetric flag supermanifold $\mathbf{\Pi F}^{n|n}_{k|k}$ of type $k=(k_1,\ldots,k_r)$ in $\mathbb{C}^{n|n}$ is a certain subsupermanifold in $\mathbf{F}^{n|n}_{k|k}$. Let us give an explicite description of these supermanifolds in terms of charts and local coordinates (see also [Man, V1, V3]).

Let us take two non-negative integers $m,n\in\mathbb{Z}$ and two sets of non-negative integers

$$k=(k_1,\ldots,k_r),\quad \text{and}\quad l=(l_1,\ldots,l_r)$$

such that (1) holds. The underlying space of the supermanifold $\mathbf{F}^{m|n}_{k|l}$ is the product $\mathbf{F}^m_k\times\mathbf{F}^n_l$ of two manifolds of flags of type $k=(k_1,\ldots,k_r)$ and $l=(l_1,\ldots,l_r)$ in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. For any $s=1,\ldots,r$ let us fix two subsets

$$I_{s\bar 0}\subset\{1,\ldots,k_{s-1}\}\quad\text{and}\quad I_{s\bar 1}\subset\{1,\ldots,l_{s-1}\},$$

where $k_0=m$ and $l_0=n$, such that $|I_{s\bar 0}|=k_s$, and $|I_{s\bar 1}|=l_s$. We set $I_s=(I_{s\bar 0},I_{s\bar 1})$ and $I=(I_1,\ldots,I_r)$. Let us assign the following $(k_{s-1}+l_{s-1})\times(k_s+l_s)$-matrix

$$Z_{I_s}=\begin{pmatrix} X_s & \Xi_s\\ \mathrm{H}_s & Y_s\end{pmatrix},\quad s=1,\ldots,r, \tag{2}$$

to any I_s. Here we assume that $X_s=(x^s_{ij})\in\mathrm{Mat}_{k_{s-1}\times k_s}(\mathbb{C})$, $Y_s=(y^s_{ij})\in\mathrm{Mat}_{l_{s-1}\times l_s}(\mathbb{C})$, and elements of the matrices $\Xi_s=(\xi^s_{ij})$, $\mathrm{H}_s=(\eta^s_{ij})$ are odd. We also assume that Z_{I_s} contains the identity submatrix $E_{k_s+l_s}$ of size $(k_s+l_s)\times(k_s+l_s)$ in the lines with numbers $i\in I_{s\bar 0}$ and $k_{s-1}+i,\ i\in I_{s\bar 1}$. For example in case

$$I_{s\bar 0}=\{k_{s-1}-k_s+1,\ldots,k_{s-1}\}\quad\text{and}\quad I_{s\bar 1}=\{l_{s-1}-l_s+1,\ldots,l_{s-1}\}$$

the matrix Z_{I_s} has the following form:

$$Z_{I_1}=\begin{pmatrix} X_s & \Xi_s\\ E_{k_s} & 0\\ \mathrm{H}_s & Y_s\\ 0 & E_{k_s}\end{pmatrix}.$$

(For simplisity of notation we use here the same letters X_s, Y_s, Ξ_s and H_s as in (2).)

We see that the sets $I_{\bar 0}=(I_{1\bar 0},\ldots,I_{r\bar 0})$ and $I_{\bar 1}=(I_{1\bar 1},\ldots,I_{r\bar 1})$ determine the charts $U_{I_{\bar 0}}$ and $V_{I_{\bar 1}}$ on the flag manifolds $\mathbf{F}^m_k$ and $\mathbf{F}^n_l$, respectively. We can take the non-trivial elements (i.e., those not contained in the identity submatrix) from X_s and Y_s as local coordinates in $U_{I_{\bar 0}}$ and $U_{I_{\bar 1}}$, respectively. Summing up, we defined an atlas

$$\{U_I=U_{I_{\bar 0}}\times U_{I_{\bar 1}}\}\quad\text{on}\quad \mathbf{F}^m_k\times\mathbf{F}^n_l$$

with chards parametrized by $I=(I_s)$. In addition the sets $I_{\bar 0}$ and $I_{\bar 1}$ determine the superdomain $\mathcal{U}_I$ with underlying space U_I and with even and odd coordinates x^s_{ij}, y^s_{ij} and ξ^s_{ij}, η^s_{ij}, respectively. (As above we assume that x^s_{ij}, y^s_{ij}, ξ^s_{ij} and η^s_{ij} are non-trivial. That is they are not contained in the identity submatrix.) Let us define the transition functions between two superdomains corresponding to $I=(I_s)$ and $J=(J_s)$ by the following formulas:

$$Z_{J_1}=Z_{I_1}C^{-1}_{I_1J_1},\quad Z_{J_s}=C_{I_{s-1}J_{s-1}}Z_{I_s}C^{-1}_{I_sJ_s},\quad s\geq 2. \tag{3}$$

Here $C_{I_sJ_s}$ is an invertible submatrix in Z_{I_s} that coinsists of lines with numbers $i \in J_{s\bar{0}}$ and $k_{s-1}+i$, where $i \in J_{s\bar{1}}$. In other words, we choose the matrix $C_{I_sJ_s}$ in such a way that Z_{J_s} contains the identity submatrix $E_{k_s+l_s}$ in lines with numbers $i \in J_{s\bar{0}}$ and $k_{s-1}+i$, where $i \in J_{s\bar{1}}$. These charts and transition functions define a supermanifold that we denote by $\mathbf{F}^{m|n}_{k|l}$. This supermanifold we will call the *supermanifold of flags* of type $k|l$. In case $r=1$ this supermanifold is called the *super-Grassmannian* and is denoted by $\mathbf{Gr}_{m|n,k|l}$ (see also [Oni, Man]).

Let us take $n \in \mathbb{N}$ and $k=(k_1,\dots,k_r)$, such that

$$0<k_1<\ldots<k_r<n.$$

We will define the *supermanifold of* Π*-symmetric flags* $\mathbf{\Pi F}^{n|n}_{k|k}$ *of type* k in $\mathbb{C}^{n|n}$ as a certain subsupermanifold in $\mathbf{F}^{n|n}_{k|k}$. The underlying space of $\mathbf{\Pi F}^{n|n}_{k|k}$ is the diagonal in $\mathbf{F}^n_k \times \mathbf{F}^n_k$, that is clearly isomorphic to $\mathbf{F}^n_k$. For any $s=1,\dots,r$ we fix a set $I_{s\bar{0}} = I_{s\bar{1}} \subset \{1,\dots,k_{s-1}\}$, where $|I_{s\bar{0}}| = k_s$ and $k_0=n$. Consider the chart on $\mathbf{F}^{n|n}_{k|k}$ corresponding to $I=(I_s)$, where $I_s=(I_{s\bar{0}}, I_{s\bar{0}})$. Such charts cover the diagonal in $\mathbf{F}^n_k \times \mathbf{F}^n_k$. Let us define the subsupermanifold of Π-symmetric flags in these charts by the equations $X_s=Y_s$, $\Xi_s=\mathrm{H}_s$. It is easy to see that these equations are well-defined with respect to the transition functions (3). The coordinate matrices in this case have the following form

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ \Xi_s & X_s \end{pmatrix}, \quad s=1,\dots,r. \tag{4}$$

(Compare with (2).) As above even and odd local coordinates on $\mathbf{\Pi F}^{n|n}_{k|k}$ are non-trivial elements from X_s and Ξ_s, respectively. The transition functions between two charts are defined again by formulas (3). We can consider the supermanifold $\mathbf{\Pi F}^{n|n}_{k|k}$ as the "set of fixed-point" of a certain odd involution Π in $\mathbb{C}^{n|n}$ (see [Man]). In case $r=1$, the supermanifold of Π-symmetric flags is called also the Π-*symmetric super-Grassmannian.* We will denote it by $\mathbf{\Pi Gr}_{n|n,k|k}$.

Let $\mathcal{M}=(\mathcal{M}_0,\mathcal{O}_{\mathcal{M}})$ be a complex-analytic supermanifold. Denote by $\mathcal{T}=\mathcal{D}er\,(\mathcal{O}_{\mathcal{M}})$ the tangent sheaf or the sheaf of vector fields on $\mathcal{M}$. It is a sheaf of Lie superalgebras with respect to the multiplication $[X,Y]=YX-(-1)^{p(X)p(Y)}XY$. The global sections of $\mathcal{T}$ are called *holomorphic vector fields* on $\mathcal{M}$. They form a complex Lie superalgebra that we will denote by $\mathfrak{v}(\mathcal{M})$. This Lie superalgebra is finite dimensional if $\mathcal{M}_0$ is compact. The goal of this paper is to compute the Lie superalgebra $\mathfrak{v}(\mathcal{M})$ in the case when $\mathcal{M}$ is a supermanifold of Π-symmetric flags of type k in $\mathbb{C}^{n|n}$.

We denote by $\mathfrak{q}_n(\mathbb{C})$ the Lie subsuperalgebra in $\mathfrak{gl}_{n|n}(\mathbb{C})$ that coinsists of the following marices:

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad \text{where} \quad A,B \in \mathfrak{gl}_n(\mathbb{C}).$$

Denote by $\mathrm{Q}_n(\mathbb{C})$ the Lie supergroup of $\mathfrak{q}_n(\mathbb{C})$. In [Man] an action of $\mathrm{Q}_n(\mathbb{C})$ on the supermanifold $\mathbf{\Pi F}^{n|n}_{k|k}$ was defined. In our coordinates this action is given by the following formulas:

$$\begin{aligned} &(L,(Z_{I_1},\dots,Z_{I_r})) \longmapsto (\tilde{Z}_{J_1},\dots,\tilde{Z}_{J_r}), \quad \text{where} \\ &L \in \mathrm{Q}_n(\mathbb{C}), \quad \tilde{Z}_{J_1} = LZ_{I_1}C_1^{-1}, \quad \tilde{Z}_{J_s} = C_{s-1}Z_{I_s}C_s^{-1}. \end{aligned} \tag{5}$$

Here C_1 is an invertible submatrix in LZ_{I_1} that consists of lines with numbers i and $n+i$, where $i \in J_1$; and $C_s,\ s \geq 2$, is an invertible submatrix in $C_{s-1}Z_{I_s}$ that consists of lines with numbers i and $k_{s-1}+i$, where $i \in J_s$. This Lie supergroup action induces a Lie superalgebra homomorphism

$$\mu : \mathfrak{q}_n(\mathbb{C}) \to \mathfrak{v}(\mathbf{\Pi F}_{k|k}^{n|n}).$$

In case $r = 1$ in [Oni, Proposition 5.5] it was proven that $\operatorname{Ker} \mu = \langle E_{2n} \rangle$, where E_{2n} is the identity matrix of size $2n$. In general case $r > 1$ the proof is similar. Hence, μ induces an injective homomorphism of Lie superalgebras $\mathfrak{q}_n(\mathbb{C})/\langle E_{2n} \rangle \to \mathfrak{v}(\mathbf{\Pi F}_{k|k}^{n|n})$. We will show that with one exception this homomorphism is an isomorphism.

3 About superbundles

Recall that a *morphism* of a complex-analytic supermanifold $\mathcal{M}$ to a complex-analytic supermanifold $\mathcal{N}$ is a pair $f = (f_0, f^*)$, where $f_0 : \mathcal{M}_0 \to \mathcal{N}_0$ is a holomorphic map and $f^* : \mathcal{O}_{\mathcal{N}} \to (f_0)_*(\mathcal{O}_{\mathcal{M}})$ is a homomorphism of sheaves of superalgebras.

Definition. We say that a *superbundle* with fiber $\mathcal{S}$, base $\mathcal{B}$, total space $\mathcal{M}$ and projection $p = (p_0, p^*) : \mathcal{M} \to \mathcal{B}$ is given if there exists an open covering $\{U_i\}$ on $\mathcal{B}_0$ and isomorphisms $\psi_i : (p_0^{-1}(U_i), \mathcal{O}_{\mathcal{M}}) \to (U_i, \mathcal{O}_{\mathcal{B}}) \times \mathcal{S}$ such that the following diagram is commutative:

$$\begin{tikzcd}
(p_0^{-1}(U_i), \mathcal{O}_{\mathcal{M}}) \arrow[r, "\psi_i"'] \arrow[d, "p"'] & (U_i, \mathcal{O}_{\mathcal{B}}) \times \mathcal{S} \arrow[d, "pr"] \\
(U_i, \mathcal{O}_{\mathcal{B}}) \arrow[r, "id"'] & (U_i, \mathcal{O}_{\mathcal{B}})
\end{tikzcd}$$

where pr is the natural projection.

Remark. From the form of transition functions (3) it follows that for $r > 1$ the supermanifold $\mathbf{\Pi F}_{k|k}^{n|n}$ is a superbundle with base $\mathbf{\Pi Gr}_{n|n,k_1|k_1}$ and fiber $\mathbf{\Pi F}_{k'|k'}^{k_1|k_1}$, where $k' = (k_2, \ldots, k_r)$. In local coordinates the projection p is given by

$$(Z_1, Z_2, \ldots Z_n) \longmapsto (Z_1).$$

Moreover, the formulas (5) tell us that the projection p is equivariant with respect to the action of the supergroup $\mathrm{Q}_n(\mathbb{C})$ on $\mathbf{\Pi F}_{k|k}^{n|n}$ and $\mathbf{\Pi Gr}_{n|n,k_1|k_1}$.

Let $p = (p_0, p^*) : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called *projectable* with respect to p, if there exists a vector field $v_1 \in \mathfrak{v}(\mathcal{N})$ such that

$$p^*(v_1(f)) = v(p^*(f)) \quad \text{for all} \quad f \in \mathcal{O}_{\mathcal{N}}.$$

In this case we say that v *is projected into* v_1. Projectable vector fields form a Lie subsuperalgebra $\overline{\mathfrak{v}}(\mathcal{M})$ in $\mathfrak{v}(\mathcal{M})$. In case if p is a projection of a superbundle, the homomorphism $p^* : \mathcal{O}_{\mathcal{N}} \to p_*(\mathcal{O}_{\mathcal{M}})$ is injective. Hence, any projectable vector field v is projected into unique vector field $v_1 = \mathcal{P}(v)$. The map

$$\mathcal{P} : \overline{\mathfrak{v}}(\mathcal{M}) \to \mathfrak{v}(\mathcal{N}), \quad v \mapsto v_1$$

is a homomorphism of Lie superalgebras. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called *vertical*, if $\mathcal{P}(v) = 0$. Vertical vector fields form an ideal $\operatorname{Ker}\mathcal{P}$ in $\overline{\mathfrak{v}}(\mathcal{M})$.

We will need the following proposition proved in [B].

Proposition 1. *Let $p : \mathcal{M} \to \mathcal{B}$ be the projection of a superbundle with fiber $\mathcal{S}$. If $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, then any holomorphic vector field from $\mathfrak{v}(\mathcal{M})$ is projectable with respect to p.*

For any superbundle $p : \mathcal{M} \to \mathcal{B}$ with fiber $\mathcal{S}$ we define the sheaf $\mathcal{W}$ on $\mathcal{B}_0$ in the following way. We asign to any open set $U \subset \mathcal{B}_0$ the set of all vertical vector fields on the supermanifold $(p_0^{-1}(U), \mathcal{O}_{\mathcal{M}})$. In [V1] the following statement was proven.

Proposition 2. *Assume that $\mathcal{S}_0$ is compact. Then $\mathcal{W}$ is a localy free sheaf of $\mathcal{O}_{\mathcal{B}}$-modules and $\dim \mathcal{W} = \dim \mathfrak{v}(\mathcal{S})$. The Lie algebra $\mathcal{W}(\mathcal{B}_0)$ coincides with the ideal of all vertical vector fields in $\mathfrak{v}(\mathcal{M})$.*

Let us describe the corresponding to $\mathcal{W}$ graded sheaf as in [V1]. Consider the following filtration in $\mathcal{O}_{\mathcal{B}}$

$$\mathcal{O}_{\mathcal{B}} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \ldots$$

where $\mathcal{J}$ is the sheaf of ideals in $\mathcal{O}_{\mathcal{B}}$ generated by odd elements. We have the corresponding graded sheaf of superalgebras

$$\tilde{\mathcal{O}}_{\mathcal{B}} = \bigoplus_{p \geq 0} (\tilde{\mathcal{O}}_{\mathcal{B}})_p,$$

where $(\tilde{\mathcal{O}}_{\mathcal{B}})_p = \mathcal{J}^p/\mathcal{J}^{p+1}$. Putting $\mathcal{W}_{(p)} = \mathcal{J}^p\mathcal{W}$ we get the following filtration in $\mathcal{W}$:

$$\mathcal{W} = \mathcal{W}_{(0)} \supset \mathcal{W}_{(1)} \supset \ldots. \qquad (6)$$

We define the $\mathbb{Z}$-graded sheaf of $\mathcal{F}_{\mathcal{B}_0}$-modules by

$$\tilde{\mathcal{W}} = \bigoplus_{p \geq 0} \tilde{\mathcal{W}}_p, \quad \text{where} \quad \tilde{\mathcal{W}}_p = \mathcal{W}_{(p)}/\mathcal{W}_{(p+1)}, \qquad (7)$$

where $\mathcal{F}_{\mathcal{B}_0}$ is the structure sheaf of the underlying space $\mathcal{B}_0$. The $\mathbb{Z}_2$-grading in $\mathcal{W}_{(p)}$ induces the $\mathbb{Z}_2$-grading in $\tilde{\mathcal{W}}_p$. Note that the natural map $\mathcal{W}_{(p)} \to \tilde{\mathcal{W}}_p$ is even.

4 Functions on Π-symmetric flag supermanifolds

In this section we show that the superbundle described in Section 2, that is the Π-symmetric flag supermanifold, satisfies conditions of Proposition 1. Holomorphic functions on other flag supermanifolds was considered in [V3].

Lemma 1. *Let $\mathcal{M}$ be a superbundle with base $\mathcal{B}$ and fiber $\mathcal{S}$. Assume that $\mathcal{O}_{\mathcal{B}}(\mathcal{B}_0) = \mathbb{C}$ and $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$. Then $\mathcal{O}_{\mathcal{M}}(\mathcal{M}_0) = \mathbb{C}$.*

In the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})_{\bar{0}} \simeq \mathfrak{gl}_n(\mathbb{C})$ we fix the following Cartan subalgebra:

$$\mathfrak{t} = \{\operatorname{diag}(\mu_1, \ldots, \mu_n)\},$$

the following system of positive roots:

$$\Delta^+ = \{\mu_i - \mu_j,\ i < j\}$$

and the following system of simple roots:

$$\Phi = \{\alpha_1, ..., \alpha_{n-1}\}, \ \ \alpha_i = \mu_i - \mu_{i+1}.$$

Denote by $\mathfrak{t}^*(\mathbb{R})$ a real subspace in $\mathfrak{t}^*$ spaned by μ_j. Consider the scalar product $(\, , \,)$ in $\mathfrak{t}^*(\mathbb{R})$ such that the vectors μ_j form an orthonormal basis. An element $\gamma \in \mathfrak{t}^*(\mathbb{R})$ is called *dominant* if $(\gamma, \alpha) \geq 0$ for all $\alpha \in \Delta^+$.

We need the Borel-Weyl-Bott Theorem (see for example [A] for details). Let $G \simeq \mathrm{GL}_n(\mathbb{C})$ be the underlying space of $\mathrm{Q}_n(\mathbb{C})$, P be a parabolic subgroup in G and R be the reductive part of P. Assume that $\mathbf{E}_\varphi \to G/P$ is the homogeneous vector bundle corresponding to a representation φ of P in $E = (\mathbf{E}_\varphi)_P$. Denote by $\mathcal{E}_\varphi$ the sheaf of holomorphic section of this vector bundle.

Theorem 1. [Borel-Weyl-Bott]. *Assume that the representation $\varphi : P \to GL(E)$ is completely reducible and $\lambda_1, ..., \lambda_s$ are highest weights of $\varphi|R$. Then the G-module $H^0(G/P, \mathcal{E}_\varphi)$ is isomorphic to the sum of irruducible G-modules with highest weights $\lambda_{i_1}, ..., \lambda_{i_t}$, where λ_{i_a} are dominant highest weights.*

The main result of this section is the following theorem.

Theorem 2. *Let $\mathcal{M} = \mathbf{\Pi F}^{n|n}_{k|k}$, then $\mathcal{O}_\mathcal{M}(\mathcal{M}_0) = \mathbb{C}$.*

Proof. First consider the case $r = 1$. This is $\mathcal{M} = \mathbf{\Pi Gr}_{n|n,k|k}$. Let us prove that $\tilde{\mathcal{O}}_\mathcal{M}(\mathcal{M}_0) = \mathbb{C}$, where $\tilde{\mathcal{O}}_\mathcal{M}$ is defined as in the previous section. We use the Borel-Weyl-Bott Theorem. The manifold $\mathcal{M}_0 = \mathbf{Gr}_{n,k}$ is isomorphic to G/H, where $G = \mathrm{GL}_n(\mathbb{C})$ and

$$H = \left\{ \left(\begin{array}{cc} A & 0 \\ C & B \end{array} \right) \middle| \ A \in \mathrm{GL}_{n-k}(\mathbb{C}), \ \ B \in \mathrm{GL}_k(\mathbb{C}) \right\}. \tag{8}$$

The reductive part R of H is given by the equation $C = 0$. It is isomorphic to $\mathrm{GL}_{n-k}(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C})$. Let ρ_1 and ρ_2 be the standard representations of $\mathrm{GL}_{n-k}(\mathbb{C})$ and $\mathrm{GL}_k(\mathbb{C})$, respectively. Denote by $\mathbf{E} \to \mathcal{M}_0$ the holomorphic vector bundle that is determined by the localy free sheaf $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$, where $\mathcal{J}$ is the sheaf of ideals generated by odd elements in $\mathcal{O}_\mathcal{M}$. In [Oni, Proposition 5.2] it was proven that $\mathbf{E}$ is a homogeneous vector bundle corresponding to the representation φ of H such that

$$\varphi|R = \rho_1^* \otimes \rho_2.$$

Since $(\tilde{\mathcal{O}}_\mathcal{M})_p \simeq \bigwedge^p \mathcal{E}$, we have to find the vector space of global sections of $\bigwedge^p \mathbf{E}$. This vector bundle corresponds to the representation $\bigwedge^p \varphi = \bigwedge^p (\rho_1^* \otimes \rho_2)$. We need to find dominat weights of $\bigwedge^p \varphi$.

Let us choose the following Cartan subalgebra

$$\mathfrak{t} = \{\mathrm{diag}(\mu_1, \ldots, \mu_n)\}$$

in $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{Lie}(G)$. For $p > 0$ any weight of this representation has the following form:

$$\Lambda = -\mu_{i_1} - \cdots - \mu_{i_p} + \mu_{j_1} + \cdots + \mu_{j_p},$$

where

$$1 \leq i_1, \ldots, i_p \leq n-k \quad \text{and} \quad n-k+1 \leq j_1, \ldots, j_p \leq n.$$

For $p = 0$ the highest weight is equal to 0. The weight $\Lambda = 0$ is clearly dominant. For $p > 0$ assume that $\mu^1 = \mu_{i_a}$, where $i_a = \max\{i_1, \dots i_p\}$, and $\mu^2 = \mu_{j_b}$, where $j_b = \min\{j_1, \dots, j_p\}$. Then, $(\Lambda, \mu^1 - \mu^2) < 0$. Therefore, the weight Λ is not dominant and we have

$$(\tilde{\mathcal{O}}_{\mathcal{M}})_0(\mathcal{M}_0) = \mathbb{C} \quad \text{and} \quad (\tilde{\mathcal{O}}_{\mathcal{M}})_p(\mathcal{M}_0) = \{0\} \quad \text{for} \quad p > 0.$$

Now let us compute the space of global holomorphic functions $\mathcal{O}_{\mathcal{M}}(\mathcal{M}_0)$. Clearly, $\mathcal{J}^p(\mathcal{M}_0) = \{0\}$ for large p. For $p \geq 0$ we have an exact sequence

$$0 \to \mathcal{J}^{p+1}(\mathcal{M}_0) \longrightarrow \mathcal{J}^p(\mathcal{M}_0) \longrightarrow (\tilde{\mathcal{O}}_{\mathcal{M}})_p(\mathcal{M}_0).$$

By induction we see that $\mathcal{J}^p(\mathcal{M}_0) = \{0\}$ for $p > 0$. Hence, for $p = 0$ our exact sequence has the form:

$$0 \to \mathcal{J}^0(\mathcal{M}_0) = \mathcal{O}_{\mathcal{M}}(\mathcal{M}_0) \longrightarrow \mathbb{C}.$$

Note that on any supermanifold we have constant functions. Hence, $\mathcal{O}_{\mathcal{M}}(\mathcal{M}_0) = \mathcal{J}^0(\mathcal{M}_0) = \mathbb{C}$. Using Lemma 1 and induction, we get the result.□

5 Vector fields on Π-symmetric flag supermanifolds

The Lie superalgebra of holomorphic vector fields on Π-symmetric super-Grassmannian $\mathbf{\Pi Gr}_{n|n,k|k}$ was computed in [Oni].

Theorem 3. *Let $\mathcal{M} = \mathbf{\Pi Gr}_{n|n,k|k}$ and $(n, k) \neq (2, 1)$. Then*

$$\mathfrak{v}(\mathcal{M}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n} \rangle.$$

Assume that $\mathcal{M} = \mathbf{\Pi Gr}_{2|2,1|1}$. Then

$$\mathfrak{v}(\mathcal{M}) \simeq \mathfrak{g} \niplus \langle z \rangle.$$

Here $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-graded Lie superalgebra defined in the following way.

$$\mathfrak{g}_{-1} = V, \quad \mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{C}), \quad \mathfrak{g}_1 = \langle \mathrm{d} \rangle,$$

where $V = \mathfrak{sl}_2(\mathbb{C})$ is the adjoint $\mathfrak{sl}_2(\mathbb{C})$-module, $[\mathfrak{g}_0, \mathfrak{g}_1] = \{0\}$ and $[\mathrm{d}, -]$ maps identicaly $\mathfrak{g}_{-1}$ to $\mathfrak{g}_0$. Here z is the grading operator of the $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$.

Proof. The first part of the proof was given in [Oni, Theorem 5.2]. The second part follows from [Oni, Theorem 4.2]. The proof is complete.□

Remark. For completeness we describe the more general Onishchik result in our particular case $\mathcal{M} = \mathbf{\Pi Gr}_{2|2,1|1}$. In local chart on $\mathbf{\Pi Gr}_{2|2,1|1}$

$$\begin{pmatrix} x & \xi \\ 1 & 0 \\ \xi & x \\ 0 & 1 \end{pmatrix} \tag{9}$$

the Lie superalgebra of vector fields $\mathfrak{v}(\mathbf{\Pi Gr}_{2|2,1|1})$ has the following form:

$$\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi}, x\frac{\partial}{\partial \xi}, x^2\frac{\partial}{\partial \xi}\rangle, \quad \mathfrak{g}_0 = \langle \frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} + 2x\xi\frac{\partial}{\partial \xi}, x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}\rangle,$$
$$\mathrm{d} = \xi\frac{\partial}{\partial x}, \quad z = \xi\frac{\partial}{\partial \xi}.$$

We will need another description of the Lie superalgebra of vector fields on $\mathbf{\Pi Gr}_{2|2,1|1}$. We have

$$\mathfrak{v}(\mathbf{\Pi Gr}_{2|2,1|1}) \simeq \mathfrak{q}_2(\mathbb{C})/\langle E_4\rangle \oplus \langle z\rangle$$

as $\mathfrak{g}_0$-modules. More precisely, in the local chart (9) the isomorphism is given by

$$(\mathfrak{q}_2(\mathbb{C})/\langle E_4\rangle)_{\bar 0} \simeq \mathfrak{g}_0 = \langle \frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} + 2x\xi\frac{\partial}{\partial \xi}, x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}\rangle,$$
$$(\mathfrak{q}_2(\mathbb{C})/\langle E_4\rangle)_{\bar 1} \simeq \langle \frac{\partial}{\partial \xi}, x\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial \xi}, \xi\frac{\partial}{\partial x}\rangle,$$
$$z = \xi\frac{\partial}{\partial \xi}.$$

From now on we use the following notations:

$$\mathcal{M} = \mathbf{\Pi F}^{n|n}_{k|k}, \quad \mathcal{B} = \mathbf{\Pi Gr}_{n|n,k_1|k_1}, \quad \mathcal{S} = \mathbf{\Pi F}^{k_1|k_1}_{k'|k'},$$

where $k' = (k_2, \ldots, k_r)$. We also assume that $r > 1$. By Proposition 1 and Theorem 2 the projection of the superbundle $\mathcal{M} \to \mathcal{B}$ determines the homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{B}).$$

This projection is equivariant. Hence for the natural Lie superalgebra homomorphisms $\mu : \mathfrak{q}_n(\mathbb{C}) \to \mathfrak{v}(\mathcal{M})$ and $\mu_{\mathcal{B}} : \mathfrak{q}_n(\mathbb{C}) \to \mathfrak{v}(\mathcal{B})$ we have

$$\mu_{\mathcal{B}} = \mathcal{P} \circ \mu.$$

Note that for $r > 1$ the base $\mathcal{B}$ cannot be isomorphic to $\mathbf{\Pi Gr}_{2|2,1|1}$. Therefore, by Theorem 3, the homomorphisms $\mu_{\mathcal{B}}$ and hence the homomorphism $\mathcal{P}$ is surjective. We will prove that $\mathcal{P}$ is injective. Hence,

$$\mu = \mathcal{P}^{-1} \circ \mu_{\mathcal{B}}$$

is surjective and

$$\mathfrak{v}(\mathcal{M}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n}\rangle.$$

Let us study $\operatorname{Ker}\mathcal{P} \subset \mathfrak{v}(\mathcal{M})$. In previous sections we constructed a localy free sheaf $\tilde{\mathcal{W}}$ on $\mathcal{B}_0$. We have a natural action of $G = \mathrm{GL}_n(\mathbb{C})$ on the sheaf $\mathcal{W}$ that preserves the filtration (6) and induces the action on the sheaf $\tilde{\mathcal{W}}$. Hence, the vector bundle $\mathbf{W}_0 \to \mathcal{B}_0$ corresponding to $\tilde{\mathcal{W}}_0$ is homogeneous. We use notations from the proof of Theorem 2. Let us compute the representation of $H \subset G$ in the fiber of $\mathbf{W}_0$ over the point $o = H \in \mathcal{B}_0$. We will identify $(\mathbf{W}_0)_o$ with the Lie superalgebra of vector fields $\mathfrak{v}(\mathcal{S})$ on $\mathcal{S}$.

Consider a local chart that contains o on the Π-symmetric super-Grassmannian $\mathcal{B}$. For example we can take the chart corresponding to $I_{1\bar{0}} = \{n-k_1+1, \dots, n\}$. The coordinate matrix (4) in this case has the following form

$$Z_{I_1} = \begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ \Xi_1 & X_1 \\ 0 & E_{k_1} \end{pmatrix}. \tag{10}$$

Let us choose an atlas of $\mathcal{M}$ in a neighborhood of o defined by certain $I_{s\bar{0}}$, $s = 2, \dots, r$, see (4). In notations (8) and (10) the group H acts on Z_{I_1} in the following way:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ C & B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & C & B \end{pmatrix} Z_{I_1} = \begin{pmatrix} AX_1 & A\Xi_1 \\ CX_1 + B & C\Xi_1 \\ A\Xi_1 & AX_1 \\ C\Xi_1 & CX_1 + B \end{pmatrix}.$$

Hence, for Z_{I_2} we have

$$\begin{aligned} &\begin{pmatrix} CX_1 + B & C\Xi_1 \\ C\Xi_1 & CX_1 + B \end{pmatrix} \begin{pmatrix} X_2 & \Xi_2 \\ \Xi_2 & X_2 \end{pmatrix} = \\ &= \begin{pmatrix} BX_2 + CX_1X_2 + C\Xi_1\Xi_2 & B\Xi_2 + CX_1\Xi_2 + C\Xi_1X_2 \\ B\Xi_2 + CX_1\Xi_2 + C\Xi_1X_2 & BX_2 + CX_1X_2 + C\Xi_1\Xi_2 \end{pmatrix}. \end{aligned} \tag{11}$$

Note that the local coordinates Z_{I_s}, $s \geq 2$, determine the local coordinate on $\mathcal{S}$. To obtain the action of H in the fiber $(\mathbf{W_0})_\mathbf{o}$ in these coordinates we put $X_1 = 0$, $\Xi_1 = 0$ in (11) and modify Z_{I_s}, $s \geq 3$, accordingly. We see that the nilradical of H and the subgroup $\mathrm{GL}_{n-k_1}(\mathbb{C})$ of R act trivially on $\mathcal{S}$ and that the subgroup $\mathrm{GL}_{k_1}(\mathbb{C}) \subset R$ acts in the natural way. This means that H acts as the even part of the Lie supergroup $Q_{k_1}(\mathbb{C})$ on Π-symmetric flag supermanifold $\mathcal{S}$, see (5).

Furthermore, by induction we assume that

$$\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{q}_{k_1}(\mathbb{C})/\langle E_{2n} \rangle \quad \text{or} \quad \mathfrak{v}(\mathcal{S}) \simeq \mathfrak{q}_2(\mathbb{C})/\langle E_4 \rangle \supsetplus \langle z \rangle.$$

Then the induced action of $\mathrm{GL}_{k_1}(\mathbb{C})$ on $\mathfrak{v}(\mathcal{S})$ coinsides with the adjoint action of the even part of $Q_{k_1}(\mathbb{C})$. Standard computations lead to the following lemma, where we denote by Ad_{k_1} the adjoint representation of $\mathrm{GL}_{k_1}(\mathbb{C})$ on $\mathfrak{sl}_{k_1}(\mathbb{C})$ and 1 is the one dimensional trivial representation of $\mathrm{GL}_{k_1}(\mathbb{C})$.

Lemma 2. *The representation ψ of H in the fiber $(\mathbf{W}_0)_o = \mathfrak{v}(\mathcal{S})$ is completely reducible. If $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{q}_{k_1}(\mathbb{C})/\langle E_{2n} \rangle$, then*

$$\psi|_{\mathfrak{v}(\mathcal{S})_{\bar{0}}} = \mathrm{Ad}_{k_1}, \ \ \psi|_{\mathfrak{v}(\mathcal{S})_{\bar{1}}} = \mathrm{Ad}_{k_1} + 1. \tag{12}$$

If $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{q}_2(\mathbb{C})/\langle E_4 \rangle \supsetplus \langle z \rangle$, then

$$\psi|_{\mathfrak{v}(\mathcal{S})_{\bar{0}}} = \mathrm{Ad}_{k_1} + 1, \ \ \psi|_{\mathfrak{v}(\mathcal{S})_{\bar{1}}} = \mathrm{Ad}_{k_1} + 1. \tag{13}$$

Further we will use the chart on $\mathbf{\Pi F}^{n|n}_{k|k}$ defined by $I_{s\bar{0}}$, where $I_{1\bar{0}}$ is as above, and

$$I_{s\bar{0}} = \{k_{s-1} - k_s + 1, \dots, k_{s-1}\}$$

for $s \geq 2$. The coordinate matrix of this chart have the following form

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ E_{k_s} & 0 \\ \Xi_s & X_s \\ 0 & E_{k_s} \end{pmatrix},$$

where again the local coordinate are $X_s = (x^s_{ij})$ and $\Xi_s = (\xi^s_{ij})$. We denote this chart by $\mathcal{U}$.

Lemma 3. *The following vector fields in* $\mathcal{U}$

$$\frac{\partial}{\partial x^1_{ij}}, \ \frac{\partial}{\partial \xi^1_{ij}}, \ u_{ij} + \frac{\partial}{\partial x^2_{ij}}, \ v_{ij} + \frac{\partial}{\partial \xi^2_{ij}}$$

are fundamental. This is they are induced by the natural action of $\mathrm{Q}_n(\mathbb{C})$ *on* $\mathcal{M}$. *Here* u_{ij} *and* v_{ij} *are vector field that depend only on coordinates from* Z_{I_1}.

Proof. Let us prove this statement for example for the vector field $\frac{\partial}{\partial x^1_{11}}$. This vector field corresponds to the one-parameter subgroup $\exp(tE_{1,n-k_1+1})$. Indeed, the action of this subgroup is given by

$$\begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ \Xi_1 & X_1 \\ 0 & E_{k_1} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{X}_1 & \Xi_1 \\ E_{k_1} & 0 \\ \Xi_1 & \tilde{X}_1 \\ 0 & E_{k_1} \end{pmatrix} \quad \text{and} \quad Z_{I_s} \mapsto Z_{I_s}, \ s \geq 2,$$

where

$$\tilde{X}_1 = \begin{pmatrix} t + x^1_{11} & \dots & x^1_{1k_1} \\ \vdots & \ddots & \vdots \\ x^1_{n-k_1,1} & \dots & x^1_{n-k_1,k_1} \end{pmatrix}. \square$$

Let us choose a basis (v_q) of $\mathfrak{v}(\mathcal{S})$. In [V1] it was proven that any holomorphic vertical vector field on $\mathcal{M}$ can be written uniquely in the form

$$w = \sum_q f_q v_q, \tag{14}$$

where f_q are holomorphic functions on $\mathcal{U}$ depending only on coordinates from Z_{I_1}. Further, we will need the following lemma:

Lemma 4. *Assume that* $\operatorname{Ker} \mathcal{P} \neq \{0\}$. *Then there exists a vector field* $w \in \operatorname{Ker} \mathcal{P} \setminus \{0\}$, *such that* $w = \sum_q f_q v_q$, *where* f_q *are holomorphic functions depending only on even coordinates from* Z_{I_1}.

Proof. Assume that in (14) there is a non-trivial vector field w such that a function f_q depends for example on ξ^1_{ij}. Then $w = \xi^1_{ij} w' + w''$, where w' and w'' are vertical vector fields and their coefficients (14) do not depend on ξ^1_{ij}, and $w' \neq 0$. Using Lemma 3 and the fact that $\operatorname{Ker} \mathcal{P}$ is an ideal in $\mathfrak{v}(\mathcal{M})$, we see that

$$w' = [w, \frac{\partial}{\partial \xi^1_{ij}}] \in \operatorname{Ker} \mathcal{P}.$$

Hence, we can exclude all odd coordinates ξ^1_{ij}. $\square$

Corollary. We have

$$(\mathrm{Ker}\,\mathcal{P} \neq \{0\}) \Longrightarrow (\mathcal{W}_0(\mathcal{B}_0) \neq \{0\}).$$

We will need the following well-known construction for holomorphic homogeneous vector bundles. Let us take any homogeneous vector bundle $\mathbf{E}$ over G/H and $x_0 = H$. Assume that $v_{x_0} \in \mathbf{E}_{x_0}$ is an H-invariant. We can construct the G-invariant section of $\mathbf{E}$ corresponding to v_{x_0} in the following way. We set

$$v_{x_1} := g \cdot v_{x_0} \in \mathbf{E}_{x_1}, \quad \text{where} \quad x_1 = gx_0.$$

Clearly this definition does not depend on $g \in G$ such that $x_1 = gx_0$. Indeed, assume that $x_1 = g_1 x_0$ and $x_1 = g_2 x_0$. Then $g_2 = g_1 h$, where $h \in H$. Hence,

$$g_2(v_{x_0}) = (g_1 h)(v_{x_0}) = g_1(v_{x_0}).$$

We use this construction to express locally the G-invariant section of $\mathbf{W}_0$ corresponding to an H-invariant in $(\mathbf{W}_0)_{x_0}$. Let us take the following element $g \in G$:

$$g = \begin{pmatrix} E_{n-k_1} & A \\ 0 & E_{k_1} \end{pmatrix} \times \begin{pmatrix} E_{n-k_1} & A \\ 0 & E_{k_1} \end{pmatrix},$$

where A is any complex matrix. Then g acts on $\mathbf{\Pi F}^{n|n}_{k|k}$ in the following way:

$$\begin{pmatrix} E_{n-k_1} & A & 0 & 0 \\ 0 & E_{k_1} & 0 & 0 \\ 0 & 0 & E_{n-k_1} & A \\ 0 & 0 & 0 & E_{k_1} \end{pmatrix} \begin{pmatrix} X^1 & \Xi^1 \\ E_{k_1} & 0 \\ \Xi^1 & X^1 \\ 0 & E_{k_1} \end{pmatrix} = \begin{pmatrix} X^1 + A & \Xi^1 \\ E_{k_1} & 0 \\ \Xi^1 & X^1 + A \\ 0 & E_{k_1} \end{pmatrix}, \tag{15}$$
$$Z_{I_s} = Z_{I_s}, \quad s > 1.$$

We see that $x_0 = H$ has coordinates $X_1 = \Xi_1 = 0$. Clearly, $\{x_1 \mid x_1 = gx_0\}$ is an open set in $\mathcal{B}_0$. Moreover, element g does not modify fiber coordinates. Therefore, the corresponding to an H-invariant $v_{x_0} \in (\mathbf{W}_0)_{x_0}$ section v is the constant section $x_1 \mapsto v_{x_0}$ over the open set $\{x_1 \mid x_1 = gx_0\}$.

Theorem 4. *Assume that $r > 1$ and $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{q}_{k_1}(\mathbb{C})/\langle E_{2k_1}\rangle$. Then* $\mathrm{Ker}\,\mathcal{P} = \{0\}$ *and*

$$\mathfrak{v}(\mathbf{\Pi F}^{n|n}_{k|k}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n}\rangle.$$

Proof. First let us compute the vector space of global sections of $\mathbf{W}_0$. As in Theorem 2, we use the Borel-Weyl-Bott Theorem. The representation ψ of H in $(\mathbf{W}_0)_o$ is described in Lemma 2. From (12) it follows that the highest weights of ψ have the form:

$$\mu_{n-k_1+1} - \mu_n \quad (\times 2) \quad \text{and} \quad 0.$$

The first highest weight is not dominant because by definition of Π-symmetric flag supermanifolds $k_1 < n$. The second highest weight is clearly dominant. Therefore, the vector space of global sections of $\mathbf{W}_0$ is the irreducible G-module with highest weight 0. Therefore, $\mathcal{W}_0(\mathcal{B}_0) \simeq \mathbb{C}$.

Let v_o be the H-invariant element form $\mathfrak{v}(\mathcal{S})$. It is defined by the following one-paremeter subsupergroup

$$\beta(\tau) = \begin{pmatrix} E_{k_1} & \tau E_{k_1} \\ \tau E_{k_1} & E_{k_1} \end{pmatrix}$$

in the Lie supergroup Q_{k_1}. In our chart we have

$$v_o = 2\sum_{ij} \xi_{ij}^2 \frac{\partial}{\partial x_{ij}^2} + u, \tag{16}$$

where u is a vector field depending only on coordinates from Z_{I_s}, $s \geq 3$. Above we have seen that the corresponding to v_o global section is a constant section. Therefore, the unique global section v of $\mathbf{W}_0$ in our chart has also the form (16).

Assume that $\operatorname{Ker}\mathcal{P} \neq \{0\}$ and $w \in \operatorname{Ker}\mathcal{P}/\{0\}$ is as in Lemma 4. Clearly the vector fields w and $\alpha(w)$, where $\alpha : \mathcal{W} \to \mathcal{W}_0$, have the same form in our chart. Therefore, $w = av$ for some $a \in \mathbb{C}$. Furthermore,

$$\alpha([w, v_{ij} + \frac{\partial}{\partial \xi_{ij}^2}]) = 2a\frac{\partial}{\partial x_{ij}^2}.$$

The commutator of these vector fields is an even vector field and it belongs to $\operatorname{Ker}\mathcal{P}$. Hence,

$$\alpha([w, v_{ij} + \frac{\partial}{\partial \xi_{ij}^2}]) = 0,$$

because $\mathbf{W}_0$ has no global even sections. Therefore, $a = 0$ and the proof is complete.$\Box$

Now consider the case when the fiber of superbundle $\mathcal{M}$ is isomorphic to $\mathbf{\Pi Gr}_{2|2,1|1}$.

Theorem 5. *Assume that* $r = 2$ *and* $\mathcal{S} = \mathbf{\Pi Gr}_{2|2,1|1}$. *Then*

$$\mathfrak{v}(\mathbf{\Pi F}^{n|n}_{2,1|2,1}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n}\rangle.$$

Proof. As in Theorem 4, let us compute the space of global sections of $\mathbf{W}_0$. From (13) it follows that the highest weights of ψ have the form:

$$\mu_{n-k_1+1} - \mu_n \quad (\times 2) \quad \text{and} \quad 0 \quad (\times 2).$$

By the Borel-Weyl-Bott Theorem we get

$$\mathcal{W}_0(\mathcal{B}_0)_{\bar{0}} \simeq \mathcal{W}_0(\mathcal{B}_0)_{\bar{1}} \simeq \mathbb{C}.$$

A basic section of $\mathcal{W}_0(\mathcal{B}_0)_{\bar{1}}$ was obtained in Theorem 4. It has the form $v = 2\xi_{11}^2 \frac{\partial}{\partial x_{11}^2}$ in our case. Furthermore, we can take $z = \xi_{11}^2 \frac{\partial}{\partial \xi_{11}^2}$ as a basic element of 1-dimensional vector subspace in $\mathfrak{v}(\mathcal{S})_{\bar{0}}$ corresponding to the trivial representation 1. Again in our local chart the unique even section of $\mathcal{W}_0(\mathcal{B}_0)_{\bar{0}}$ has locally the form $s = \xi_{11}^2 \frac{\partial}{\partial \xi_{11}^2}$.

Let us take the vector field w as in Lemma 4. The vector fields $\alpha(w)$ and w have the same form in our chart. Hence, $w = av + bs$, where $a, b \in \mathbb{C}$. Furthermore,

$$\alpha([w, v_{11} + \frac{\partial}{\partial \xi_{11}^2}]) = \alpha(2a\frac{\partial}{\partial x_{11}^2} + b\frac{\partial}{\partial \xi_{11}^2}) = 2a\frac{\partial}{\partial x_{11}^2} + b\frac{\partial}{\partial \xi_{11}^2} \in \mathcal{W}_0(\mathcal{B}_0) = \langle v, z\rangle.$$

In other words, $2a\frac{\partial}{\partial x_{11}^2} + b\frac{\partial}{\partial \xi_{11}^2}$ must be a linear combination of v and z. Hence, $a = b = 0$, and the proof is compete.$\Box$

By induction we get our main result:

Theorem 6. *Assume that $r > 1$. Then $\mathfrak{v}(\mathbf{\Pi F}_{k|k}^{n|n}) \simeq \mathfrak{q}_n(\mathbb{C})/\langle E_{2n}\rangle$.*

6 The automorphism supergroup of $\mathbf{\Pi F}_{k|k}^{n|n}$

Our main result has infinitesimal nature. However we can determine the connected component of the automorphism supergroup of $\mathbf{\Pi F}_{k|k}^{n|n}$. Let us discuss this statement in details.

Let us take a complex-analytic supermanifold $\mathcal{M}$ with a compact underlying space $\mathcal{M}_0$. Then the Lie superalgebra of holomorphic vector fields $\mathfrak{v}(\mathcal{M})$ is finite dimensional. Denote by $\mathcal{A}ut(\mathcal{M})_{\bar{0}}$ the Lie group of even global automorphisms of $\mathcal{M}$. (The fact that $\mathcal{A}ut(\mathcal{M})_{\bar{0}}$ is a complex-analytic Lie group with the Lie algebra $\mathfrak{v}(\mathcal{M})_{\bar{0}}$ was proven in [BK].) Moreover, we have a natural holomorphic action of $\mathcal{A}ut(\mathcal{M})_{\bar{0}}$ on $\mathcal{M}$ (see [BK]) and hence on $\mathfrak{v}(\mathcal{M})$. Therefore, the pair $(\mathcal{A}ut(\mathcal{M})_{\bar{0}}, \mathfrak{v}(\mathcal{M}))$ is a super Harish-Chandra pair. (See citeViLieSupergroup for the definition of a super Harish-Chandra pair.) Using the equivalence of complex super Harish-Chandra pairs and complex Lie supergroups obtained in [V5] we determine the complex Lie supergroup $\mathcal{A}ut(\mathcal{M})$. We call this Lie supergroup the *automorphism supergroup* of $\mathcal{M}$.

Consider the case $\mathcal{M} = \mathbf{\Pi F}_{k|k}^{n|n}$. Above we described a holomorphic action of $\mathrm{GL}_n(\mathbb{C}) = \mathrm{Q}_n(\mathbb{C})_{\bar{0}}$ on $\mathcal{M}$. In other words we have a homomorphism of Lie groups

$$\mathrm{Q}_n(\mathbb{C})_{\bar{0}} \to \mathcal{A}ut(\mathcal{M})_{\bar{0}}. \tag{17}$$

This homomorphism induces (almost always, see Theorems 3 and 6) the isomorphism of Lie algebras $\mathfrak{q}_n(\mathbb{C})_{\bar{0}}/\langle E_{2n}\rangle$ and $\mathfrak{v}(\mathcal{M})_{\bar{0}}$, see Theorems 3 and 6. In Section 2 we have seen that the kernel of the homomorphism (17) is equal to $\{\alpha E_{2n}\}$, where $\alpha \neq 0$, or to the center $\mathcal{Z}(\mathrm{Q}_n(\mathbb{C})_{\bar{0}})$ of $\mathrm{Q}_n(\mathbb{C})_{\bar{0}}$. Therefore, the connected component of the automorphism supergroup $\mathcal{A}ut^0(\mathbf{\Pi F}_{k|k}^{n|n})$ is determined by the super Harish-Chandra pair

$$(\mathrm{Q}_n(\mathbb{C})_{\bar{0}}/\mathcal{Z}(\mathrm{Q}_n(\mathbb{C})_{\bar{0}}), \mathfrak{q}_n(\mathbb{C})/\langle E_{2n}\rangle).$$

In other words,

$$\mathcal{A}ut^0(\mathbf{\Pi F}_{k|k}^{n|n}) \simeq \mathrm{Q}_n(\mathbb{C})/\mathcal{Z}(\mathrm{Q}_n(\mathbb{C})).$$

In case $\mathcal{M} = \mathbf{\Pi Gr}_{2|2,1|1}$, the connected component of the automorphism supergroup $\mathcal{A}ut^0(\mathbf{\Pi Gr}_{2|2,1|1})$ is given by the following super Harish-Chandra pair:

$$(\mathrm{Q}_2(\mathbb{C})_{\bar{0}}/\mathcal{Z}(\mathrm{Q}_2(\mathbb{C})_{\bar{0}}) \times \mathbb{C}^*, \mathfrak{v}(\mathbf{\Pi Gr}_{2|2,1|1})),$$

see Theorem 3 for a description of $\mathfrak{v}(\mathbf{\Pi Gr}_{2|2,1|1})$.

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